

## NOTE

# A Dyson Constant Term Orthogonality Relation

Kevin W. J. Kadell

*Department of Mathematics, Arizona State University, Tempe, Arizona 85287*

*Communicated by George Andrews*

Received April 27, 1998



provided by Elsevier - Publisher Connector

constant term in  $\prod_{1 \leq i < j \leq n} (1 - t_i/t_j)^{a_i} (1 - t_j/t_i)^{a_j}$  is the multinomial coefficient  $(a_1 + \dots + a_n)!/a_1! \dots a_n!$ . These and other results suggest that there should exist Dyson polynomials which generalize the Macdonald polynomials. © 2000 Academic Press

## 1. INTRODUCTION AND SUMMARY

Let  $n \geq 2$  and  $a_1, \dots, a_n \geq 0$ . We set

$$f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \prod_{1 \leq i < j \leq n} \left(1 - \frac{t_i}{t_j}\right)^{a_i} \left(1 - \frac{t_j}{t_i}\right)^{a_j}. \quad (1.1)$$

Let  $[w]f$  denote the coefficient of the monomial  $w$  in the Laurent expansion of  $f$ . Dyson [Dy1] conjectured the constant term identity

$$[1] f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}, \quad (1.2)$$

which was proven independently by Gunson [Gu1] and Wilson [Wi1]. Good [Go1] gave a short proof of (1.2) using the identity

$$1 = \sum_{j=1}^n \prod_{\substack{i \neq j \\ i=1}}^n \left(1 - \frac{t_j}{t_i}\right)^{-1} \quad (1.3)$$



Let  $r \geq 1$ . We denote the compositions of  $r$  by

$$(w_1, \dots, w_n) \sim r \Leftrightarrow r = w_1 + \dots + w_n, \quad w_1, \dots, w_n \geq 0. \quad (1.4)$$

The complete homogeneous symmetric function of order  $r$  is given by

$$h_r(s_1, \dots, s_n) = \sum_{(w_1, \dots, w_n) \sim r} \prod_{i=1}^n s_i^{w_i}. \quad (1.5)$$

Let  $\tilde{s}^a$  denote the set of variables  $(s, \dots, s)$  in which the variable  $s$  is repeated exactly  $a$  times. Thus  $(\tilde{t}_1^{a_1}, \dots, \tilde{t}_n^{a_n})$  denotes the set of variables which is obtained by repeating  $t_i$  exactly  $a_i$  times for  $1 \leq i \leq n$ .

We set

$$\begin{aligned} f_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n; t_1, \dots, t_n) \\ = \prod_{i=1}^n t_i^{-w_i} h_r(\tilde{t}_1^{a_1}, \dots, \tilde{t}_n^{a_n}) f_n(a_1, \dots, a_n; t_1, \dots, t_n) \end{aligned} \quad (1.6)$$

and use capital letters to denote the constant term

$$F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n) = [1] f_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n; t_1, \dots, t_n). \quad (1.7)$$

The following theorem is our main result.

**THEOREM 1.** *Let  $n \geq 2$ ,  $a_1, \dots, a_n \geq 0$ ,  $r \geq 1$  and  $(w_1, \dots, w_n) \sim r$ . Then we have*

$$w_1, \dots, w_n < r \Rightarrow F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n) = 0 \quad (1.8)$$

and

$$F_{n,r}^{-r, 0, \dots, 0}(a_1, \dots, a_n) = \frac{a_1}{(1 + \sum_{v=2}^n a_v)_r} \frac{(a_1 + \dots + a_n + r - 1)!}{a_1! \dots a_n!}. \quad (1.9)$$

These and other results suggest that there should exist Dyson polynomials which generalize the Macdonald polynomials.

In Section 2, we use Good's proof [Gol] to establish the constant term orthogonality relation Theorem 1 (1.8) and the normalization (1.9).

In Section 3, we give a conjectured  $q$ -analogue of Theorem 1 which has been verified by computer for  $n=3$ ,  $0 \leq a_1, a_2, a_3 \leq 4$ , and summarize the prospects for generalizing the Macdonald polynomials.

## 2. A PROOF OF THEOREM 1

In this section, we use Good's proof [Go1] to establish the constant term orthogonality relation Theorem 1 (1.8) and the normalization (1.9).

We proceed by induction on  $N = a_1 + \cdots + a_n$  and  $n$  where  $a_1, \dots, a_n \geq 0$  and  $n \geq 2$ . When  $n = 1$  the constant term orthogonality relation (1.8) is vacuous. Let  $u \geq 1$ . Observe that there is a one-to-one correspondence between compositions  $(w_1, \dots, w_u) \sim r$  of  $r$  into  $u$  parts and subsets  $A \subseteq [1, r + u - 1]$  with  $u - 1$  elements given by

$$A = \{m - 1 + a_1 + \cdots + a_m \mid 1 \leq m \leq u - 1\}. \quad (2.1)$$

Hence

$$h_r(\tilde{t}_1^{a_1}) = \binom{r + a_1 - 1}{a_1 - 1} t_1^r = \frac{(a_1)_r}{r!} t_1^r. \quad (2.2)$$

Thus when  $n = 1$  we see that the normalization (1.9) reduces to the fact that the empty product is equal to one.

Let  $n \geq 2$  and assume that the constant term orthogonality relation Theorem 1 (1.8) and the normalization (1.9) hold with  $n$  replaced by  $n - 1$ .

Observe that if  $a_m = 0$  where  $1 \leq m \leq n$ , then the variable  $t_m$  does not occur to a positive power in any of the terms in the expansion of  $h_r(\tilde{t}_1^{a_1}, \dots, \tilde{t}_n^{a_n}) f_n(a_1, \dots, a_n; t_1, \dots, t_n)$ . Thus we have the boundary conditions

$$a_m = 0 \text{ and } w_m > 0 \text{ where } 1 \leq m \leq n \Rightarrow F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n) = 0 \quad (2.3)$$

and

$$\begin{aligned} a_m = w_m = 0 \text{ where } 1 \leq m \leq n &\Rightarrow F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n) \\ &= F_{n-1,r}^{-w_1, \dots, -w_{m-1}, -w_{m+1}, \dots, -w_n}(a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_n). \end{aligned} \quad (2.4)$$

Let  $u \geq 1$ . Observe that

$$h_r(s_1, \dots, s_u) = h_r(s_1, \dots, s_{u-1}) + s_u h_{r-1}(s_1, \dots, s_u). \quad (2.5)$$

Let  $1 \leq m \leq n$ . Applying (2.5) to one of the  $a_m$  copies of  $t_m$  gives

$$\begin{aligned} h_r(\tilde{t}_1^{a_1}, \dots, \tilde{t}_n^{a_n}) &= h_r(\tilde{t}_1^{a_1}, \dots, \tilde{t}_{m-1}^{a_{m-1}}, \tilde{t}_m^{a_m-1}, \tilde{t}_{m+1}^{a_{m+1}}, \dots, \tilde{t}_n^{a_n}) \\ &\quad + t_m h_{r-1}(\tilde{t}_1^{a_1}, \dots, \tilde{t}_n^{a_n}). \end{aligned} \quad (2.6)$$

We have the Vandermonde determinant

$$\Delta_n(t_1, \dots, t_n) = \prod_{1 \leq i < j \leq n} (t_i - t_j) = \det |t_j^{n-i}|_{n \times n}. \quad (2.7)$$

Let  $1 \leq z \leq n-1$ . Then we have

$$0 = \det |t_j^{n-i\chi(i < n) + z\chi(i = n)}|_{n \times n} \quad (2.8)$$

since rows  $n-z$  and  $n$  are equal. Expanding along the bottom row, we obtain

$$\begin{aligned} 0 &= \sum_{m=1}^n (-1)^{n+m} t_m^z \det |t_j^{n-i}|_{i \neq n, j \neq m} \\ &= \sum_{m=1}^n (-1)^{n+m} t_m^z \prod_{\substack{j=1 \\ j \neq m}}^n t_j \prod_{\substack{1 \leq i < j \leq n \\ i \neq m \neq j}} (t_i - t_j) \\ &= \Delta_n(t_1, \dots, t_n) \sum_{m=1}^n (-1)^{n+m} t_m^z \prod_{i=1}^{m-1} \frac{t_i}{(t_i - t_m)} \prod_{j=m+1}^n \frac{t_j}{(t_m - t_j)}. \end{aligned} \quad (2.9)$$

Dividing (2.9) by  $\Delta_n(t_1, \dots, t_n)$  and performing some algebra, we have

$$0 = \sum_{m=1}^n t_m^z \prod_{\substack{i=1 \\ i \neq m}}^n \left(1 - \frac{t_m}{t_i}\right)^{-1}. \quad (2.10)$$

We may write (2.10) as

$$0 = \sum_{m=1}^n t_m^z f_n(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n; t_1, \dots, t_n). \quad (2.11)$$

Observe that Good's identity (1.3) gives

$$f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \sum_{m=1}^n f_n(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n; t_1, \dots, t_n). \quad (2.12)$$

Using Good's identity in the form (2.12) and the case  $z=1$  of (2.11), we have

$$\begin{aligned}
& f_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n; t_1, \dots, t_n) \\
&= \prod_{i=1}^n t_i^{-w_i} h_r(\tilde{t}_1^{a_1}, \dots, \tilde{t}_n^{a_n}) \\
&\quad \times \sum_{m=1}^n f_n(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n; t_1, \dots, t_n) \\
&= \prod_{i=1}^n t_i^{-w_i} \sum_{m=1}^n h_r(t_1^{a_1}, \dots, \tilde{t}_{m-1}^{a_{m-1}}, \tilde{t}_m^{a_m-1}, t_{m+1}^{a_{m+1}}, \dots, t_n^{a_n}) \\
&\quad \times f_n(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n; t_1, \dots, t_n) \\
&\quad + \prod_{i=1}^n t_i^{-w_i} h_{r-1}(\tilde{t}_1^{a_1}, \dots, \tilde{t}_n^{a_n}) \\
&\quad \times \sum_{m=1}^n t_m f_n(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n; t_1, \dots, t_n) \\
&= \sum_{m=1}^n f_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n; t_1, \dots, t_n). \quad (2.13)
\end{aligned}$$

Thus the function  $f_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n; t_1, \dots, t_n)$  satisfies the functional equation (2.12).

Extracting the constant term from (2.13), we have

$$\begin{aligned}
& F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n) \\
&= \sum_{m=1}^n F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n). \quad (2.14)
\end{aligned}$$

The constant term orthogonality relation Theorem 1 (1.8) follows using the functional equation (2.14) and the boundary conditions (2.3) and (2.4).

We leave the reader to use the distributive law to prove the following lemma which allows us to generate solutions of the functional equation (2.14) of Good's proof.

**LEMMA 2.** *Let  $m, \ell \geq 1$ ,  $n = m + \ell - 1$ ,  $1 \leq h \leq m$ , and let  $\mathcal{A}_m(a_1, \dots, a_m)$  and  $\mathcal{B}_\ell(a_1, \dots, a_\ell)$  satisfy the functional equation (2.14) with  $n = m$  and  $n = \ell$ , respectively. Then the function*

$$\begin{aligned}
& \mathcal{F}_n(a_1, \dots, a_n) \\
&= \mathcal{A}_m(a_1, \dots, a_{h-1}, a_h + \dots + a_{h+\ell-1}, a_{h+\ell}, \dots, a_n) \mathcal{B}_\ell(a_h, \dots, a_{h+\ell-1}) \quad (2.15)
\end{aligned}$$

*satisfies the functional equation (2.14).*

Take  $m = h = 2$ ,  $\ell = n - 1$ ,  $\mathcal{A}_2(\alpha, \beta) = \alpha(\alpha + \beta + r - 1)!/\alpha!(\beta + r)! = (\alpha + \beta + r - 1)!/(\alpha - 1)!(\beta + r)!$  and  $\mathcal{B}_{n-1}(a_2, \dots, a_n) = (a_2 + \dots + a_n)!/a_2! \dots a_n!$  in Lemma 2 (2.15) and set  $\alpha = a_1$ ,  $\beta = a_2 + \dots + a_n$ . We see that the function on the right side of (1.9) satisfies the functional equation (2.14).

The normalization Theorem 1 (1.9) follows using the functional equation (2.14) and the boundary conditions (2.3) and (2.4), completing the proof of Theorem 1.

### 3. A CONJECTURED $q$ -ANALOGUE OF THEOREM 1

In this section, we give a conjectured  $q$ -analogue of Theorem 1 which has been verified by computer for  $n = 3$ ,  $0 \leq a_1, a_2, a_3 \leq 4$ , and summarize the prospects for the Dyson polynomials.

Let  $|q| < 1$  and set  $(x; q)_m = \prod_{i=1}^m (1 - xq^{i-1})$ . Following Andrews [An1], we set

$${}_q f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \prod_{1 \leq i < j \leq n} \left( \frac{t_i}{t_j}; q \right)_{a_i} \left( q \frac{t_j}{t_i}; q \right)_{a_j}. \quad (3.1)$$

Andrews [An1] conjectured a  $q$ -analogue of the Dyson constant term identity which has been proven by Zeilberger and Bressoud [ZB1]. This is given by the following theorem.

**THEOREM 3** (Zeilberger and Bressoud [ZB1])

$$[1] {}_q f_n(a_1, \dots, a_n; t_1, \dots, t_n) = \frac{(q; q)_{a_1 + \dots + a_n}}{(q; q)_{a_1} \dots (q; q)_{a_n}}. \quad (3.2)$$

We set

$$\begin{aligned} & {}_q f_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n; t_1, \dots, t_n) \\ &= \prod_{i=1}^n t_i^{-w_i} h_r(t_1, \dots, q^{a_1-1} t_1, \dots, t_n, \dots, q^{a_n-1} t_n) {}_q f_n(a_1, \dots, a_n; t_1, \dots, t_n) \end{aligned} \quad (3.3)$$

and use capital letters to denote the constant term

$${}_q F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n) = [1] {}_q f_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n; t_1, \dots, t_n). \quad (3.4)$$

The following conjecture, which has been extensively verified by computer, provides a  $q$ -analogue of Theorem 1.

CONJECTURE 4. Let  $n \geq 2$ ,  $a_1, \dots, a_n \geq 0$ ,  $r \geq 1$  and  $(w_1, \dots, w_n) \sim r$ . Then we have

$$w_1, \dots, w_n < r \Rightarrow {}_q F_{n,r}^{-w_1, \dots, -w_n}(a_1, \dots, a_n) = 0 \quad (3.5)$$

and

$${}_q F_{n,r}^{-r, 0, \dots, 0}(a_1, \dots, a_n) = \frac{(1 - q^{a_1})}{(q^{a_2 + \dots + a_n}; q)_r} \frac{(q; q)_{a_1 + \dots + a_n + r - 1}}{(q; q)_{a_1} \dots (q; q)_{a_n}}. \quad (3.6)$$

Macdonald [Ma1, Chap. VI] essentially proves Conjecture 4 (3.5) and (3.6) when  $a_1 = \dots = a_n$ . See also Stanley [St1] and Kadell [Ka2]. The results of [Ka2, Ka3] suggest that there should exist Dyson polynomials which generalize the Macdonald polynomials and use the Zeilberger–Bressoud Theorem 3 (3.2) as the weight function.

## REFERENCES

- [An1] G. E. Andrews, Problems and prospects for basic hypergeometric functions, in “Theory and Applications of Special Functions” (R. A. Askey, Ed.), Academic Press, New York, 1975.
- [Dyl] F. J. Dyson, Statistical theory of the energy levels of complex systems, I, *J. Math. Phys.* **3** (1962), 140–156.
- [Gol] I. J. Good, Short proof of a conjecture of Dyson, *J. Math. Phys.* **11** (1970), 1884.
- [Gul] J. Gunson, Proof of a conjecture by Dyson in the statistical theory of energy levels, *J. Math. Phys.* **3** (1962), 752–753.
- [Ka1] K. W. J. Kadell, The Selberg–Jack symmetric functions, *Adv. Math.* **130** (1997), 33–101.
- [Ka2] K. W. J. Kadell, Aomoto’s machine and the Dyson constant term identity, *Methods Appl. Anal.* **5** (1998), 335–350.
- [Ka3] K. W. J. Kadell, The Dyson polynomials, preprint.
- [Ma1] I. G. Macdonald, “Symmetric Functions and Hall Polynomials,” 2nd ed., Oxford Univ. Press, Oxford, UK, 1995.
- [St1] R. P. Stanley, Some combinatorial properties of Jack symmetric functions, *Adv. Math.* **77** (1989), 76–115.
- [Wil] K. Wilson, Proof of a conjecture by Dyson, *J. Math. Phys.* **3** (1962), 1040–1043.
- [ZB1] D. Zeilberger and D. M. Bressoud, A proof of Andrews’  $q$ -Dyson conjecture, *Discrete Math.* **54** (1985), 201–224.